ELSEVIER

Journal of Geometry and Physics 43 (2002) 341–344

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

WZW–Poisson manifolds

C. Klimčík [a], T. Strobl [b,*]

[a] *Institute de Mathématiques de Luminy, 163 Avenue de Luminy, 13288 Marseille, France*
[b] *Institut für Theoretische Physik, Friedrich Schiller Universität, Max-Wien-Platz 1, D-07743 Jena, Germany*

Received 23 April 2001; received in revised form 7 January 2002

Abstract

We observe that a term of the WZW-type can be added to the Lagrangian of the Poisson σ-model in such a way that the algebra of the first class constraints remains closed. This leads to a natural generalization of the concept of Poisson geometry. The resulting "WZW–Poisson" manifold M is characterized by a bivector Π and by a closed three-form H such that $1/2[\Pi, \Pi]_{\mathrm{Schouten}} = \langle H, \Pi \otimes \Pi \otimes \Pi \rangle$.

MSC: 70G45; 53D17; 53D55

Subj. Class.: Differential geometry

Keywords: Twisted Poisson manifolds; Dirac structures; Controlled nonassociativity

The presence of a closed three-form field combined with other geometrical data on a manifold may lead to discoveries of rich mathematical structures having deep applications in theoretical physics. As a basic example of this situation we mention the construction of the so-called WZW model in [1]. The three-form in question is the bi-invariant Cartan form on a (simple) Lie group manifold. Its presence in the σ-model action leads to conformal invariance at the quantum level and consequently to many interesting ramifications both in mathematics and physics.

This paper studies a generalization of Poisson manifolds induced naturally by the presence of closed three-forms. In other words, we shall consider triples (M, Π, H), where M is the manifold, Π the bivector, and H the closed three-form. When H vanishes, the bivector will have to satisfy the so-called Jacobi identity $[\Pi, \Pi]_{\mathrm{S}} = 0$ ($[\cdot, \cdot]_{\mathrm{S}}$ is the Schouten bracket). The manifolds with such a special Π are called Poisson and their study now represents itself a well developed discipline in geometry.

* Corresponding author.
E-mail address: pth@tpi.uni-jena.de (T. Strobl).

PII: S0393-0440(02)00027-X

It was the idea of Schaller and Strobl [2] to associate to the Poisson manifold a special dynamical system living on a two-dimensional surface and usually referred to as the Poisson σ-model. They have shown that the Jacobi identity is the condition which makes this σ-model a maximally constrained dynamical system (see below for more details). This paper is based on the generalization of this observation. We actually add to the standard Poisson σ-model a term containing the three-form field H in virtually the same way as one adds the three-form term to the action of the usual WZW model (see Eq. (9) below). Then we ask the question: which condition must Π and H fulfil so that the model (9) is a maximally constrained dynamical system? The result is Eq. (10), which can be interpreted as Jacobi identity in the background of the three-form H.

This paper provides the necessary technical details leading to Eq. (10). It was our intention to make it short and not to develop the classical theory further without controlling the concept of quantization. Indeed, we believe that the real understanding of the significance of the generalized Jacobi identity will lie in presenting (10) as a sort of a semiclassical approximation of some algebras with controlled non-associativity. The WZW–Poisson σ-model (9) should then stand in the core of the non-associative generalization of the Kontsevich formula. Although we are not able to offer here the quantization of our classical story, we do believe that the generalized σ-model (9) and the condition (10) contain the germ of structures as rich as those related to the standard Poisson geometry.

We begin by the study of the following example: given a bivector $\Pi = 1/2\Pi^{ij}\partial_i \wedge \partial_j$ and a two-form $\Omega = 1/2\Omega_{ij}\,\mathrm{d}X^i \wedge \mathrm{d}X^j$ on a manifold M, we can immediately write down the action functional

$$S[X, A] = \int_\Sigma \left(A_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X)A_i \wedge A_j + \frac{1}{2}\Omega_{ij}(X)\,\mathrm{d}X^i \wedge \mathrm{d}X^j\right). \tag{1}$$

In the story that follows, Σ will be a cylindrical world-sheet, X^i is a collection of coordinates on the target space M, and A_i is a set of 1-forms on Σ. Of course this action can be written also in a coordinate-independent way.

Introducing the standard world-sheet coordinates σ and τ (the loop and the evolution parameters, respectively) we set

$$A_i = A_{i\sigma}\,\mathrm{d}\sigma + A_{i\tau}\,\mathrm{d}\tau, \tag{2}$$

and rewrite (1) in the following form

$$S[X, A] = \int_\Sigma \mathrm{d}\sigma\,\mathrm{d}\tau[p_i\partial_\tau X^i - A_{i\tau}\phi^i]. \tag{3}$$

Here

$$p_i = A_{i\sigma} - \Omega_{ij}\partial_\sigma X^j, \qquad \phi^i = \partial_\sigma X^i + \Pi^{ij}p_j + \Pi^{ij}\Omega_{jk}\partial_\sigma X^k. \tag{4}$$

Let P be some (possibly infinite-dimensional) manifold equipped with a symplectic form $\mathrm{d}\theta$. Suppose there is a set of functions $h, \phi^\alpha, d^\alpha_\beta, c^{\alpha\beta}_\gamma$ fulfilling

$$\{h, \phi^\alpha\} = d^\alpha_\beta\phi^\beta, \qquad \{\phi^\alpha, \phi^\beta\} = c^{\alpha\beta}_\gamma\phi^\gamma, \tag{5}$$

where the indices take values in some set U and the Poisson bracket corresponds to $\mathrm{d}\theta$. To these data we associate the constrained dynamical system described by the action

$$S = \int (\theta - (h + \lambda_\alpha \phi^\alpha)\,\mathrm{d}\tau), \tag{6}$$

where λ_α is a set of Lagrange multipliers and ϕ^α are the corresponding first class constraints.

Now the question arises: for which pair Π, Ω does the action (3) define a maximally constrained dynamical system in the sense described above (i.e. the relations (5) should hold). Of course, $h = 0$, $A_{i\tau}$ play the role of the Lagrange multipliers λ_α and $\theta = \oint p_i\,\mathrm{d}X^i$.

It is simple to answer this question. The symplectic form $\mathrm{d}\theta$ has the canonical Darboux form and the calculation of the Poisson brackets is straightforward. We obtain

$$\{\phi^i(\sigma), \phi^j(\sigma')\} = -(\partial_k \Pi^{ij} + \Pi^{il}\Pi^{jm}(\mathrm{d}\Omega)_{klm})\delta(\sigma - \sigma')\phi^k(\sigma), \tag{7}$$

provided

$$\tfrac{1}{2}[\Pi, \Pi]_{\mathrm{S}} = \langle \mathrm{d}\Omega, \Pi \otimes \Pi \otimes \Pi \rangle, \tag{8}$$

holds true. The symbol $[\cdot,\cdot]_{\mathrm{S}}$ denotes the Schouten bracket and the functions $c_\gamma^{\alpha\beta}$ can be read off from (7). The contraction on the right hand side is with respect to the first, third and fifth entry of Π^3. We remark that the condition (8) is necessary and sufficient for the system of constraints following from (1) to be of the first class (cf. [3] for further details).

Our discussion can be slightly generalized. Consider the bivector Π and a *closed* 3-form on the manifold M. To these data we associate the following action

$$S[X, A] = \int_\Sigma \left(A_i \wedge \mathrm{d}X^i + \frac{1}{2}\Pi^{ij}(X) A_i \wedge A_j \right) + \int_V H. \tag{9}$$

Here V is the interior of the cylinder Σ and by H we really mean the pullback of H to V by an extension to V of the map $X^i(\sigma, \tau)$. Of course, there are the subtleties concerning the boundaries of the cylinder and the WZW term. We do not give the detailed discussion in this letter. It is a straightforward generalization of the treatment in [4], where the WZW model on the cylinder is studied from the point of view of Hamiltonian mechanics.

Note that (9) reduces to (1) for $H = \mathrm{d}\Omega$. By repeating the previous discussion, we arrive at the conclusion that the model (9) corresponds to a maximally constrained dynamical system iff

$$\frac{1}{2}[\Pi, \Pi]_{\mathrm{S}} = \langle H, \Pi \otimes \Pi \otimes \Pi \rangle. \tag{10}$$

For $H = 0$, the action (9) defines the Poisson σ-model [2,5,6] and the condition (10) says that the bivector Π satisfies the Jacobi identity. Therefore Poisson geometry could have been invented by asking the question when the model (9) (with $H = 0$) is a maximally constrained dynamical system or a topological field theory. If we do not set $H = 0$, the same logic gives a natural generalization: the concept of what one might call WZW–Poisson manifolds. We repeat that the latter is characterized by a bivector Π and a closed 3-form H such that the condition (10) holds.

It remains to understand the properties of the WZW–Poisson manifolds in more detail. It may be that there is a non-trivial intersection of this notion with the other generalizations of

Poisson geometry like quasi-Poisson manifolds [7], Dirac manifolds [8] or the manifolds leading to the non-associative generalization [9,10] of the Kontsevich expansion.

Note added: after completion of this work we became aware that the relation (10) was obtained also in [11] within a BV approach.

References

[1] E. Witten, Nonabelian bosonization in two dimensions, Commun. Math. Phys. 92 (1984) 455.

[2] P. Schaller, T. Strobl, Poisson structure induced (topological) field theories, Mod. Phys. Lett. A 9 (1994) 3129;
P. Schaller, T. Strobl, Poisson-σ-models: a generalization of 2D gravity-Yang-Mills systems, in: A.N. Sissakian, G.S. Pogosyan (Eds.), Proceedings of the International Workshop on Finite Dimensional Integrable Systems, Dubna, 1995, p. 181. hep-th/9411163;
P. Schaller, T. Strobl, Introduction to Poisson σ-Models, LNP, Vol. 469, Springer, Berlin, 1996, p. 321. hep-th/9507020.

[3] T. Strobl, in preparation.

[4] C. Klimčík, P. Ševera, Open strings and D-branes in WZNW model, Nucl. Phys. B 488 (1997) 653.

[5] N. Ikeda, Two-dimensional gravity and nonlinear gauge theory, Ann. Phys. 235 (1994) 435.

[6] A. Alekseev, P. Schaller, T. Strobl, The topological G/G WZW model in the generalized momentum representation, Phys. Rev. D 52 (1995) 7146.

[7] A. Alekseev, Y. Kosmann-Schwarzbach, E. Meinrenken, Quasi-Poisson manifolds, Math.DG/0006168.

[8] T. Courant, Dirac manifolds, Trans. AMS 319 (1990) 631.

[9] L. Cornalba, R. Schiappa, Nonassociative star product deformations for D-brane worldvolumes in curved backgrounds. hep-th/0101219.

[10] C.M. Hofman, W.K. Ma, Deformations of closed strings and topological open membranes. hep-th/0102201.

[11] J.-S. Park, Topological open p-branes. hep-th/0012141.